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An Application on an Information System via Nano Ordered Topology

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Abstract

Rough set theory is commonly used to handle uncertainty in various applications. In order to broaden its application scope, the classical rough set model based on equivalence relations, it has been extended to include an additional partial order relation. This partial order relation represents an m-nano flou set, as defined in Section 5, between rough sets and is particularly useful in determining the levels of impact that key factors have on heart failure. The primary objective of the current research is to introduce a novel approximation method based on equivalence relations and partial order relations (ordered approximation spaces), which extends Pawlak's method and investigates related results. The paper establishes the equivalence between our approach and Pawlak's approach under the condition that we have an equivalence relation and a partial order relation that satisfies the criteria required for it to be considered an equality relation. The second objective is to extend the concept of nano topology to include nano ordered topology, which involves nano increasing or decreasing topological spaces. The research indicates that incorporating nano increasing or decreasing topological spaces results in enhanced data analysis accuracy when compared to solely utilizing nano topological spaces. This observation aligns with the discussions in the referenced work by Jayalakshmi in [16]. The findings of this research have the potential to significantly impact medical research related to heart failure. Improved methods for handling uncertainty and quantifying the influence of various factors can lead to more accurate and reliable predictions and diagnoses. Ultimately, this work aims to contribute to advancements in heart failure treatment and prevention. By bridging the gap between traditional rough set theory and the nuanced intricacies of heart failure analysis, our research strives to advance our comprehension of this critical medical condition and, in turn, support progress in heart failure treatment and prevention.

Keywords: increasing (decreasing, boundary) lower (upper) approximations; nano ordered topological spaces; m-nano flou set; decision making.

1 Introduction

Rough set theory is a mathematical framework designed to address uncertainty by utilizing precise lower and upper approximation sets. These approximations precisely define the sets within the minimal or maximal rough set, respectively. Originally introduced by Pawlak [30], rough set theory extends conventional set theory to accommodate intelligent systems dealing with limited and incomplete data. Central to this theory is the concept of equivalence relations or partitions, which formalizes information granulation. Lower and upper approximations within an approximation space are essential components for representing concepts within the given space or information system. These approximations are instrumental in managing uncertainty and incomplete information, ultimately facilitating a more comprehensive comprehension of the concepts in focus. Some of them used reflexive relations [3], similarity relations [4, 34], general binary relations [20, 38], topological structures [29, 39], and coverings [41, 42]. Marei proposed some different methods based on topological structures and neighborhoods to generalize Pawlak rough sets in [22, 23]. On the other hand, Raafat [31] introduced and studied some methods based on the ideal concept and topological structures to generalize the previous methods such as[21]. Some relationships between the rough set approach [30] and the other branches studied in [36, 37].

Topologists have leveraged the concept of relations to construct a comprehensive topology that serves as a versatile mathematical framework applicable to any group interconnected through these relations. We conclude that the relations have been entered to build topological structures in a variety of fields such as in rough sets and their extensions [1, 3], rough multisets [2], decision-making problems [2, 11], medical applications [7, 8], bipolar soft ordered topology Demirtas [6], economic fields [5, 10], topological reductions of attributes for predicting of a lung cancer disease [7] and heart failure [8], biochemistry [15], computer sciences [17, 19], structure analysis [13], fuzzy soft approaches [26, 27], topological study of zeolite socony mobil-5 [32], near sets theory [24], and covering rough sets [40, 41]. In 2022, Dalkilic [5] introduced some topological structures of virtual fuzzy parameterized fuzzy soft sets and proposed some applications of his methods.

In 1965, Nachbin [28] introduced a topological ordered space by incorporating a partial order relation into the structure of a topological space. This extension allows for the consideration of ordered relationships alongside topological properties, making topological ordered spaces a generalization of traditional topological spaces. Building on this concept, Mc Cartan [25] utilized monotone neighborhoods to introduce and explore ordered separation axioms, further contributing to the study of topological ordered spaces. The extension of indiscernibility to situations with an additional partial order relation between objects is a natural progression. Significantly, according to Lemma 3.4, our approach is equivalent to Pawlak's approach in cases where we have both an equivalence relation and a partial order relation that fulfills the conditions of being the equality relation.

Topology and its applications, as discussed by Sierpinski [33] in 1956, have proven to be highly relevant and impactful in various real-life scenarios. Numerous studies [9, 12] demonstrate the practical significance of topology. An interesting development in this field is the concept of "nano-topology", which emerged from a general topology induced by Pawlak's rough set approximations [35]. This nano-topology, dependent on an equivalence relation, has found applications in various fields. To broaden its application, we aim to extend the notion of nano-topology using the proposed approximations. We will demonstrate that this generalized method is more accurate than another method previously presented by Jayalakshmi [16].

Additionally, we propose a new method for topological reduction of attributes derived from an information table. Unlike Pawlaks approach, this method considers a partial order relation in the information table, making it more generalized. In a case study conducted by Jayalakshmi [16], the identification of key risk factors contributing to heart failure was explored. These risk factors included high blood pressure, alcohol and smoking, diabetes, stress and strain, and family history of early heart attacks.

In response to this quest, we introduce a novel extension to the classical rough set model, one that transcends the confines of equivalence relations and ventures into the realm of partial order relations. The scope of our work is inherently tied to a fundamental concern - the pressing need for an enhanced understanding of the factors underpinning heart failure. Heart failure, a multifaceted medical condition, eludes simple categorization due to its intricate web of contributing factors, ranging from genetic predisposition to lifestyle choices. The classical rough set model, grounded in equivalence relations, often falls short in capturing the subtleties and nuances inherent in these relationships.

Heart failure is not merely a binary outcome; it exists along a spectrum, influenced by myriad factors with varying degrees of impact. This inherent complexity demands an advanced analytical framework capable of teasing out these nuances. In our quest to address this issue, we introduce an additional dimension to rough set theory, the "m-nano flou set." This concept, elucidated in Section 5, enables us to more accurately assess the relative impacts of key factors on heart failure.

Our research has a dual focus: firstly, to propose the novel notion of "ordered approximation spaces," which serves as the bedrock for our extended rough set model. Building upon these spaces, we introduce a pioneering generalization of Pawlak rough sets and their attendant extensions. This not only broadens the toolbox of rough set theorists but also offers a fresh perspective on data analysis in complex, uncertain scenarios. Secondly, we aim to extend the horizons of rough set theory by unveiling the concept of "nano ordered topology." This expansion introduces nano increasing and decreasing topological spaces, which we demonstrate, through rigorous methodology, significantly enhance the precision of data analysis. This discovery corroborates the insights put forth in the seminal work of Jayalakshmi in 2017 [16].

Our findings, rooted in meticulous research and experimentation, carry the potential for profound impact, particularly within the realm of medical research. Improved methods for managing uncertainty and quantifying the influence of diverse factors offer the promise of more accurate predictions and diagnoses. Ultimately, our work is positioned to contribute substantially to advancements in heart failure treatment and prevention.

The rest of this paper is organized as follows. Section 2 presents a comprehensive overview of the increasing (decreasing) set and fundamental concepts in rough set theory. Additionally, it offers a concise introduction to flou sets and nano-topology. Moving on to Section 3, we propose an innovative ordered approximation space. In Section 4, we present the concept of nano ordered topology. To illustrate this concept, we provide an example. Through theoretical analysis, we establish the monotonicity of the associated uncertainty measures, which includes the nano increasing (or decreasing) accuracy measure. In Section 5, we introduce the concept of m-nano flou sets and conduct a comprehensive comparison with other relevant method. Finally, in Section 6, we conclude our research findings.

2 Preliminaries

Definition 2.1. [18] *A partial order is a binary relation $\lesssim$ defined on a set Ψ that satisfies three properties: reflexivity, antisymmetry, and transitivity. The pair $(\Psi, \lesssim)$ constitutes a partially ordered set (POS). Additionally, the equality relation on Ψ, denoted by $\blacktriangle$, represents the set of all pairs of the form (ℓ, ℓ) for each element ℓ in Ψ.*

Definition 2.2. [28] *Consider a POS $(\Psi, \lesssim)$, where Ψ is a set, and let ℓ be an element in Ψ, while Υ is a subset of Ψ. Then:*

1. $i(\ell) = \{b \in \Psi : \ell \lesssim b\}$, *and* $d(\ell) = \{b \in \Psi : b \lesssim \ell\}$.
2. $i(\Upsilon) = \{b \in \Psi : \ell \lesssim b \text{ for some } \ell \in \Upsilon\} = \cup_{\ell \in \Upsilon}(i(\ell))$, *and*
   $d(\Upsilon) = \{b \in \Psi : b \lesssim \ell \text{ for some } \ell \in \Upsilon\} = \cup_{\ell \in \Upsilon}(d(\ell))$.

It is evident that if $i(\Upsilon) = \Upsilon$, then Υ can be considered an increasing set, and if $d(\Upsilon) = \Upsilon$, then Υ can be regarded as a decreasing set.

Definition 2.3. [30] *Consider a non-empty finite set of objects known as the universe, denoted by Ψ, and an equivalence relation $\Re$ defined on Ψ. This pair, denoted as $(\Psi, \Re)$, is referred to as an approximation space. Now, let Υ be a subset of Ψ. Then:*

1. *The lower approximation of Υ with respect to $\Re$ is the set of all objects that can confidently be classified as belonging to Υ based on $\Re$. This lower approximation is denoted by $L_{\Re}(\Upsilon)$, defined as $L_{\Re}(\Upsilon) = \bigcup \{\Re(\nu) : \Re(\nu) \subseteq \Upsilon\}$, where $\Re(\nu)$ represents the equivalence class determined by $\nu \in \Psi$.*
2. *The upper approximation of Υ with respect to $\Re$ is the set of all objects that can potentially be classified as belonging to Υ based on $\Re$. This upper approximation is denoted by $U_{\Re}(\Upsilon)$, defined as $U_{\Re}(\Upsilon) = \bigcup \{\Re(\nu) : \Re(\nu) \cap \Upsilon \neq \emptyset\}$.*
3. *The boundary of the region of Υ with respect to $\Re$ is the set of all objects that cannot be decisively classified as either belonging to Υ or not based on $\Re$. This boundary is denoted by $B_{\Re}(\Upsilon)$, defined as $B_{\Re}(\Upsilon) = U_{\Re}(\Upsilon) - L_{\Re}(\Upsilon)$.*
4. *The accuracy measure, denoted by $C_{\Re}(\Upsilon)$, represents the degree of crispness for a set Υ with respect to the equivalence relation $\Re$. It is defined as the ratio of the cardinality of the lower approximation to the cardinality of the upper approximation, given by $C_{\Re}(\Upsilon) = \dfrac{|L_{\Re}(\Upsilon)|}{|U_{\Re}(\Upsilon)|}$. If $U_{\Re}(\Upsilon) \neq \emptyset$, the set Υ is considered a crisp set when $U_{\Re}(\Upsilon) = L_{\Re}(\Upsilon)$ with respect to $\Re$; otherwise, it is considered a rough set.*

Proposition 2.1. [30] *Given an approximation space $(\Psi, \Re)$, let $\Upsilon, \Gamma \subseteq \Psi$. Then:*

1. $L_{\Re}(\Upsilon) \subseteq \Upsilon \subseteq U_{\Re}(\Upsilon)$.
2. $L_{\Re}(\emptyset) = \emptyset$ *and* $U_{\Re}(\Psi) = \Psi$.
3. $L_{\Re}(\Upsilon) \subseteq L_{\Re}(\Gamma)$ *and* $U_{\Re}(\Upsilon) \subseteq U_{\Re}(\Gamma)$ *whenever* $\Upsilon \subseteq \Gamma$.
4. $L_{\Re}(\Upsilon \cap \Gamma) \subseteq L_{\Re}(\Upsilon) \cap L_{\Re}(\Gamma)$ *and* $L_{\Re}(\Upsilon) \cup L_{\Re}(\Gamma) \subseteq L_{\Re}(\Upsilon \cup \Gamma)$.
5. $U_{\Re}(\Upsilon \cap \Gamma) \subseteq U_{\Re}(\Upsilon) \cap U_{\Re}(\Gamma)$ *and* $U_{\Re}(\Upsilon) \cup U_{\Re}(\Gamma) \subseteq U_{\Re}(\Upsilon \cup \Gamma)$.

6. $L_{\Re}(L_{\Re}(\Upsilon)) = L_{\Re}(\Upsilon)$ and $U_{\Re}(U_{\Re}(\Upsilon)) = U_{\Re}(\Upsilon)$.
7. $U_{\Re}(\Upsilon^c) = \big[L_{\Re}(\Upsilon)\big]^c$ and $L_{\Re}(\Upsilon^c) = \big[U_{\Re}(\Upsilon)\big]^c$.

Definition 2.4. [35] *Consider a universe Ψ and an equivalence relation $\Re$ on Ψ. For any subset $\Upsilon \subseteq \Psi$, the collection $\tau_{\Re}(\Upsilon) = \big\{\Psi, \emptyset, L_{\Re}(\Upsilon), U_{\Re}(\Upsilon), B_{\Re}(\Upsilon)\big\}$ is called the nano topology on Ψ. By Property 2.1, the nano topology $\tau_{\Re}(\Upsilon)$ satisfies the following axioms:*

1. *Both Ψ and $\emptyset$ are in $\tau_{\Re}(\Upsilon)$.*
2. *The union of any subcollection of $\tau_{\Re}(\Upsilon)$ is also in $\tau_{\Re}(\Upsilon)$.*
3. *The intersection of any finite subcollection of $\tau_{\Re}(\Upsilon)$ is also in $\tau_{\Re}(\Upsilon)$.*

The pair $\big(\Psi, \tau_{\Re}(\Upsilon)\big)$ is referred as a nano topological space. The elements of $\tau_{\Re}(\Upsilon)$ are called nano open sets, and their complements are called nano closed sets.

Definition 2.5. [14] *A flou set in a universe Ψ is represented by a pair (Π, χ) of subsets of Ψ, where $\Pi \subseteq \chi$. The subset Π is referred to as the center zone, χ is known as the maximal zone, and $\Pi - \chi$ is termed the flou zone.*

Definition 2.6. [14] *An m-flou set ω in a universe Ψ $(m \geq 2)$ is represented by an m-tuple $\big(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_m\big)$. The core of ω is denoted by $core(\omega) = \Pi_1$, and the hull of ω is denoted by hull $(\omega) = \Pi_m$.*

3 Ordered Approximation Spaces

In this section, we present the concept of the ordered approximation space as a generalization of Pawlak's approximation. We explore several properties and illustrate them with counterexamples for better understanding.

Definition 3.1. *Consider a universe set Ψ, an equivalence relation $\Re$ on Ψ, and a partial order relation $\lesssim$ defined on Ψ. For each $\nu \in \Psi$, we define the following concepts:*

1. *The increasing equivalence class is denoted by $I\Re(\nu)$ and is defined as $I\Re(\nu) = \cup\big\{i(\beta) : \beta \in \Psi, \nu\Re\beta\big\}$.*
2. *The decreasing equivalence class is denoted by $D\Re(\nu)$ and is defined as $D\Re(\nu) = \cup\big\{d(\beta) : \beta \in \Psi, \nu\Re\beta\big\}$.*

Lemma 3.1. *Let $\Re$ be an equivalence relation and $\lesssim$ a partial order relation on the set Ψ. For any element $\nu \in \Psi$:*

1. *$\nu \in I\Re(\nu)$ and $\nu \in D\Re(\nu)$.*
2. *$I\Re(\nu) \neq \emptyset$ and $D\Re(\nu) \neq \emptyset$.*

Proof. The veracity of statements 1 and 2 can be deduced directly from Definition 3.1. □

Remark 3.1. *If $\zeta \in I\Re(\nu)$ or $\big(D\Re(\nu)\big)$, then it may not be the case that $I\Re(\zeta) \subseteq I\Re(\nu)$ or $\big(D\Re(\zeta) \subseteq D\Re(\nu)\big)$ as explained in the following example.*

Example 3.1. *Let* $\Psi = \{\rho, \delta, \sigma, \varsigma\}$ *with* $\Re = \blacktriangle \cup \big\{(\rho, \sigma), (\sigma, \rho)\big\}$ *and* $\lesssim = \blacktriangle \cup \big\{(\rho, \delta), (\delta, \varsigma), (\rho, \varsigma)\big\}$. *Then:*

$$\begin{array}{lllll} i(\rho) = \{\rho, \delta, \varsigma\}, & d(\rho) = \{\rho\}, & \Re(\rho) = \{\rho, \sigma\}, & I\Re(\rho) = \Psi, & D\Re(\rho) = \{\rho, \sigma\}, \\ i(\delta) = \{\delta, \varsigma\}, & d(\delta) = \{\rho, \delta\}, & \Re(\delta) = \{\delta\}, & I\Re(\delta) = \{\delta, \varsigma\}, & D\Re(\delta) = \{\rho, \delta\}, \\ i(\sigma) = \{\sigma\}, & d(\sigma) = \{\sigma\}, & \Re(\sigma) = \{\rho, \sigma\}, & I\Re(\sigma) = \Psi, & D\Re(\sigma) = \{\rho, \sigma\}, \\ i(\varsigma) = \{\varsigma\}, & d(\varsigma) = \{\rho, \delta, \varsigma\}, & \Re(\varsigma) = \{\varsigma\}, & I\Re(\varsigma) = \{\varsigma\}, & D\Re(\varsigma) = \{\rho, \delta, \varsigma\}. \end{array}$$

Clear, $\rho \in D\Re(\delta)$, *but* $D\Re(\rho) \nsubseteq D\Re(\delta)$.

Remark 3.2. *In Example 3.1, if we consider* $\lesssim = \blacktriangle \cup \big\{(\varsigma, \delta), (\delta, \rho), (\varsigma, \rho)\big\}$, *then we observe that* $\rho \in I\Re(\varsigma)$, *but it is not true that* $I\Re(\rho) \nsubseteq I\Re(\varsigma)$.

Lemma 3.2. *Let* $\Re$ *be an equivalence relation and* $\lesssim$ *is a partial order relation on* Ψ, *then for each* $\nu \in \Psi$:

1. $\Re(\nu) \subseteq I\Re(\nu)$.
2. $\Re(\nu) \subseteq D\Re(\nu)$.

Proof.

1. $\Re(\nu) = \cup\big\{\beta : \beta \in \Psi, \nu\Re\beta\big\} \subseteq \cup\big\{i(\beta) : \beta \in \Psi, \nu\Re\beta\big\} = I\Re(\nu)$.
2. Similar to 1.

□

Lemma 3.3. *If* $\Re$ *is an equivalence relation, and* $\lesssim$ *is an equality relation on* Ψ, *then for each* $\nu \in \Psi$:

1. $I\Re(\nu) = \Re(\nu)$.
2. $D\Re(\nu) = \Re(\nu)$.

Proof.

1. Since $\lesssim$ is an equality relation, then $i(\beta) = \beta, \ \forall \beta \in \Psi$.
   Therefore, $I\Re(\nu) = \{i(\beta) : \beta \in \Psi, \nu\Re\beta\} = \{\beta : \beta \in \Psi, \nu\Re\beta\} = \Re(\nu)$.
2. Similar to 1.

□

Definition 3.2. *An ordered approximation space* (OAS) *is denoted by* $\big(\Psi, \Re, \lesssim\big)$, *where* Ψ *is a universe set,* $\Re$ *is an equivalence relation on* Ψ, *and* $\lesssim$ *is a partial order relation defined on* Ψ.

Definition 3.3. *Let* $\big(\Psi, \Re, \lesssim\big)$ *be an ordered approximation space. For a subset* Υ *of* Ψ, *the following concepts are defined:*

1. *The increasing lower approximation of* Υ *with respect to* $\Re$ *and* $\lesssim$ *is denoted by* $IL_{\Re}(\Upsilon)$ *and defined as:* $IL_{\Re}(\Upsilon) = \cup\big\{I\Re(\nu) : \Re(\nu) \subseteq \Upsilon\big\} \cap \Upsilon$.

2. *The decreasing lower approximation of* Υ *with respect to* $\Re$ *and* $\lesssim$ *is denoted by* $DL_{\Re}(\Upsilon)$ *and defined as:* $DL_{\Re}(\Upsilon) = \cup\big\{D\Re(\nu) : \Re(\nu) \subseteq \Upsilon\big\} \cap \Upsilon$.
3. *The increasing upper approximation of* Υ *with respect to* $\Re$ *and* $\lesssim$ *is denoted by* $IU_{\Re}(\Upsilon)$ *and defined as:* $IU_{\Re}(\Upsilon) = \big[DL_{\Re}(\Upsilon^c)\big]^c$.
4. *The decreasing upper approximation of* Υ *with respect to* $\Re$ *and* $\lesssim$ *is denoted by* $DU_{\Re}(\Upsilon)$ *and defined as:* $DU_{\Re}(\Upsilon) = \big[IL_{\Re}(\Upsilon^c)\big]^c$.
5. *The increasing boundary approximations of* Υ *with respect to* $\Re$ *and* $\lesssim$ *is denoted by* $IB_{\Re}(\Upsilon)$ *and defined as:* $IB_{\Re}(\Upsilon) = IU_{\Re}(\Upsilon) - IL_{\Re}(\Upsilon)$.
6. *The decreasing boundary approximations of* Υ *with respect to* $\Re$ *and* $\lesssim$ *is denoted by* $DB_{\Re}(\Upsilon)$ *and defined as:* $DB_{\Re}(\Upsilon) = DU_{\Re}(\Upsilon) - DL_{\Re}(\Upsilon)$.

Proposition 3.1. *Given an* OAS $(\Psi, \Re, \lesssim)$, *let* $\Upsilon, \Gamma \subseteq \Psi$. *Then:*

1. $IL_{\Re}(\Upsilon) \subseteq \Upsilon \subseteq IU_{\Re}(\Upsilon), \ \big(DL_{\Re}(\Upsilon) \subseteq \Upsilon \subseteq DU_{\Re}(\Upsilon)\big)$.
2. $IL_{\Re}(\emptyset) = \emptyset$ *and* $IU_{\Re}(\Psi) = \Psi, \ \big(DL_{\Re}(\emptyset) = \emptyset$ *and* $DU_{\Re}(\Psi) = \Psi\big)$.
3. $\Upsilon \subseteq \Gamma \Longrightarrow IL_{\Re}(\Upsilon) \subseteq IL_{\Re}(\Gamma), \ \big(\Upsilon \subseteq \Gamma \Longrightarrow DL_{\Re}(\Upsilon) \subseteq DL_{\Re}(\Gamma)\big)$.
4. $\Upsilon \subseteq \Gamma \Longrightarrow IU_{\Re}(\Upsilon) \subseteq IU_{\Re}(\Gamma), \ \big(\Upsilon \subseteq \Gamma \Longrightarrow DU_{\Re}(\Upsilon) \subseteq DU_{\Re}(\Gamma)\big)$.
5. $IL_{\Re}(\Upsilon \cap \Gamma) \subseteq IL_{\Re}(\Upsilon) \cap IL_{\Re}(\Gamma), \ \big(DL_{\Re}(\Upsilon \cap \Gamma) \subseteq DL_{\Re}(\Upsilon) \cap DL_{\Re}(\Gamma)\big)$.
6. $IL_{\Re}(\Upsilon) \cup IL_{\Re}(\Gamma) \subseteq IL_{\Re}(\Upsilon \cup \Gamma), \ \big(DL_{\Re}(\Upsilon) \cup DL_{\Re}(\Gamma) \subseteq DL_{\Re}(\Upsilon \cup \Gamma)\big)$.
7. $IU_{\Re}(\Upsilon \cap \Gamma) \subseteq IU_{\Re}(\Upsilon) \cap IU_{\Re}(\Gamma), \ \big(DU_{\Re}(\Upsilon \cap \Gamma) \subseteq DU_{\Re}(\Upsilon) \cap DU_{\Re}(\Gamma)\big)$.
8. $IU_{\Re}(\Upsilon) \cup IU_{\Re}(\Gamma) \subseteq IU_{\Re}(\Upsilon \cup \Gamma), \ \big(DU_{\Re}(\Upsilon) \cup DU_{\Re}(\Gamma) \subseteq DU_{\Re}(\Upsilon \cup \Gamma)\big)$.
9. $IL_{\Re}(IL_{\Re}(\Upsilon)) = IL_{\Re}(\Upsilon), \ \big(DL_{\Re}(DL_{\Re}(\Upsilon)) = DL_{\Re}(\Upsilon)\big)$.
10. $IU_{\Re}(IU_{\Re}(\Upsilon)) = IU_{\Re}(\Upsilon), \ \big(DU_{\Re}(DU_{\Re}(\Upsilon)) = DU_{\Re}(\Upsilon)\big)$.
11. $IL_{\Re}(IU_{\Re}(\Upsilon)) \subseteq IU_{\Re}(\Upsilon), \ \big(DL_{\Re}(DU_{\Re}(\Upsilon)) \subseteq DU_{\Re}(\Upsilon)\big)$.
12. $IL_{\Re}(\Upsilon) \subseteq IU_{\Re}(IL_{\Re}(\Upsilon)), \ \big(DL_{\Re}(\Upsilon) \subseteq DU_{\Re}(DL_{\Re}(\Upsilon))\big)$.
13. $IU_{\Re}(\Upsilon^c) = [DL_{\Re}(\Upsilon)]^c, \ \big(DU_{\Re}(\Upsilon^c) = [IL_{\Re}(\Upsilon)]^c\big)$.

Proof.

1. From Definition 3.3, it can be inferred that $IL_{\Re}(\Upsilon) \subseteq \Upsilon$ and $DL_{\Re}(\Upsilon) \subseteq \Upsilon$, then $\Upsilon^c \subseteq [DL_{\Re}(\Upsilon)]^c = IU_{\Re}(\Upsilon^c)$. So, $\Upsilon \subseteq IU_{\Re}(\Upsilon)$. Therefore, $IL_{\Re}(\Upsilon) \subseteq \Upsilon \subseteq IU_{\Re}(\Upsilon)$.
2. We can observe that $IL_{\Re}(\emptyset) = \emptyset$ since the intersection of $\emptyset$ with the union of all sets that are subsets of $\emptyset$ is itself empty. Conversely, $IU_{\Re}(\Psi) = \Psi$ as it represents the complement of $\emptyset$ (which is $DL_{\Re}(\emptyset)$) within the set Ψ.
3. $IL_{\Re}(\Upsilon) = \cup\big\{I\Re(\nu) : \Re(\nu) \subseteq \Upsilon\big\} \cap \Upsilon \subseteq \cup\big\{I\Re(\nu) : \Re(\nu) \subseteq \Gamma\big\} \cap \Gamma = IL_{\Re}(\Gamma)$.
4. If $\Upsilon \subseteq \Gamma$, then $\Gamma^c \subseteq \Upsilon^c$. This means that $DL_{\Re}(\Gamma^c) \subseteq DL_{\Re}(\Upsilon^c)$, by (3). Therefore, $[DL_{\Re}(\Upsilon^c)]^c \subseteq [DL_{\Re}(\Gamma^c)]^c$. As a result, $IU_{\Re}(\Upsilon) \subseteq IU_{\Re}(\Gamma)$.

5. $IL_{\Re}(\Upsilon\cap\Gamma) = \cup\{I\Re(\nu) : \Re(\nu) \subseteq (\Upsilon\cap\Gamma)\} \cap (\Upsilon\cap\Gamma) \subseteq \cup\{I\Re(\nu) : \Re(\nu) \subseteq \Upsilon\} \cap \Upsilon = IL_{\Re}(\Upsilon)$. Similarly, $IL_{\Re}(\Upsilon\cap\Gamma) \subseteq IL_{\Re}(\Gamma)$. Therefore, $IL_{\Re}(\Upsilon\cap\Gamma) \subseteq IL_{\Re}(\Upsilon) \cap IL_{\Re}(\Gamma)$.

6. $IL_{\Re}(\Upsilon) = \cup\{I\Re(\nu) : \Re(\nu) \subseteq \Upsilon\} \cap \Upsilon \subseteq \cup\{I\Re(\nu) : \Re(\nu) \subseteq (\Upsilon\cup\Gamma)\} \cap (\Upsilon\cup\Gamma) = IL_{\Re}(\Upsilon\cup\Gamma)$. Similarly, $IL_{\Re}(\Gamma) \subseteq IL_{\Re}(\Upsilon\cup\Gamma)$. Consequently, $IL_{\Re}(\Upsilon) \cup IL_{\Re}(\Gamma) \subseteq IL_{\Re}(\Upsilon\cup\Gamma)$.

7. $IU_{\Re}(\Upsilon\cap\Gamma) = \left[DL_{\Re}(\Upsilon\cap\Gamma)^c\right]^c = \left[DL_{\Re}(\Upsilon^c\cup\Gamma^c)\right]^c \subseteq \left[DL_{\Re}(\Upsilon^c)\cup DL_{\Re}(\Gamma^c)\right]^c = \left[DL_{\Re}(\Upsilon^c)\right]^c \cap \left[DL_{\Re}(\Gamma^c)\right]^c = IU_{\Re}(\Upsilon) \cap IU_{\Re}(\Gamma)$.

8. $IU_{\Re}(\Upsilon\cup\Gamma) = \left[DL_{\Re}(\Upsilon\cup\Gamma)^c\right]^c = \left[DL_{\Re}(\Upsilon^c\cap\Gamma^c)\right]^c \supseteq \left[DL_{\Re}(\Upsilon^c)\cap DL_{\Re}(\Gamma^c)\right]^c = \left[DL_{\Re}(\Upsilon^c)\right]^c \cup \left[DL_{\Re}(\Gamma^c)\right]^c = IU_{\Re}(\Upsilon) \cup IU_{\Re}(\Gamma)$.

9. Let $\Gamma = IL_{\Re}(\Upsilon)$ and $y \in \Gamma = \cup\{I\Re(\nu) : \Re(\nu) \subseteq \Upsilon\} \cap \Upsilon$. So, $y \in I\Re(\nu) \cap \Upsilon \subseteq I\Re(\nu)$. This implies that $y \in I\Re(\nu) \cap \Gamma$. Then, $y \in IL_{\Re}(\Gamma)$ and so $IL_{\Re}(\Upsilon) \subseteq IL_{\Re}(IL_{\Re}(\Upsilon))$. By using property (1), it can be deduced that $IL_{\Re}(IL_{\Re}(\Upsilon)) \subseteq IL_{\Re}(\Upsilon)$. Therefore, $IL_{\Re}(IL_{\Re}(\Upsilon)) = IL_{\Re}(\Upsilon)$.

10. By utilizing property (9) and the definition of $IU_{\Re}$, we can deduce that $IU_{\Re}(IU_{\Re}(\Upsilon)) = IU_{\Re}(\Upsilon)$.

11. Let $\Lambda = IU_{\Re}(\Upsilon)$, and according to property (1), $IL_{\Re}(\Lambda) \subseteq \Lambda$. This means that $IL_{\Re}(IU_{\Re}(\Upsilon)) \subseteq IU_{\Re}(\Upsilon)$.

12. According to the definition of Γ, if $\Gamma = IL_{\Re}(\Upsilon)$, property (1) implies that $\Gamma \subseteq IU_{\Re}(\Gamma)$. In other words, $IL_{\Re}(\Upsilon) \subseteq IU_{\Re}(IL_{\Re}(\Upsilon))$.

13. $\left[DL_{\Re}(\Upsilon)\right]^c = \left[IU_{\Re}(\Upsilon^c)^c\right]^c = IU_{\Re}(\Upsilon^c)$.

A corresponding proof is provided for the cases mentioned in the parentheses. □

In general, properties 11 and 12 in Proposition 3.1 do not hold in the opposite direction. The following example explains that:

Example 3.2. *Let* $\Psi = \{1, 2, ..., 8\}$ *with* $\Re = \blacktriangle \cup \{(1,4),(4,1),(1,5),(5,1),(4,5),(5,4),(3,6),(6,3)\}$ *and* $\lesssim = \blacktriangle \cup \{(1,3),(1,6),(4,3),(7,3),(4,6),(5,6),(7,6)\}$. *Then:*

1. *If* $\omega = \{1,2,7\} \subseteq \Psi$, *then* $IU_{\Re}(\omega) = \left[DL_{\Re}(\omega^c)\right]^c = \left[\{3,4,5,6,8\}\right]^c = \{1,2,7\}$, *and* $IL_{\Re}(IU_{\Re}(\omega)) = \{2,3,6,7\} \cap \{1,2,7\} = \{2,7\}$. *Therefore,* $IU_{\Re}(\omega) \nsubseteq IL_{\Re}(IU_{\Re}(\omega))$. *Also, if* $\omega = \{1,3\} \subseteq \Psi$, *then* $DU_{\Re}(\omega) = \left[IL_{\Re}(\omega^c)\right]^c = \left[\{2,6,7,8\}\right]^c = \{1,3,4,5\}$, *and* $DL_{\Re}(DU_{\Re}(\omega)) = \{1,4,5\} \cap \{1,3,4,5\} = \{1,4,5\}$. *Therefore,* $DU_{\Re}(\omega) \nsubseteq DL_{\Re}(DU_{\Re}(\omega))$.

2. *If* $\Upsilon = \{1,2,3,7\} \subseteq \Psi$, *then* $IL_{\Re}(\Upsilon) = \{2,3,7\}$, *and* $IU_{\Re}(IL_{\Re}(\Upsilon)) = IU_{\Re}(\{2,3,7\}) = \left[DL_{\Re}(\{2,3,7\})^c\right]^c = \left[\{1,4,5,8\} \cap \{1,4,5,6,8\}\right]^c = \{2,3,6,7\} \neq IL_{\Re}(\Upsilon)$. *Also, if* $\Upsilon = \{1,3,4,6,7\} \subseteq \Psi$, *then* $DL_{\Re}(\Upsilon) = \{1,3,4,6,7\}$, *and* $DU_{\Re}(DL_{\Re}(\Upsilon)) = DU_{\Re}(\{1,3,4,6,7\}) = [IL_{\Re}(\{1,3,4,6,7\})^c]^c = \left[\{2,8\} \cap \{2,5,8\}\right]^c = \{1,3,4,5,6,7\} \neq DL_{\Re}(\Upsilon)$.

Remark 3.3. *Proposition 3.1 demonstrates that the introduced approximations adhere to all properties of Pawlak approximations in the general scenario, without imposing any additional restrictions or conditions on the two relations. As a result, we can assert that our methodology represents a generalization of Pawlak's rough set theory. The following result further validates this claim.*

Lemma 3.4. *If* $\Re$ *is an equivalence relation with* $\lesssim = \blacktriangle$ *on* Ψ *and* $\Upsilon \subseteq \Psi$. *Then,* $IL_{\Re}(\Upsilon) = DL_{\Re}(\Upsilon) = L_{\Re}(\Upsilon)$ *and* $IU_{\Re}(\Upsilon) = DU_{\Re}(\Upsilon) = U_{\Re}(\Upsilon)$.

Proof. By Lemma 3.3, the proof is evident. □

4 Nano Ordered Topological Spaces

In this section, we introduce the concept of nano ordered topology. To illustrate this concept, we provide an example. Through theoretical analysis, we demonstrate the monotonicity of the corresponding uncertainty measures, including the nano increasing (or decreasing) accuracy measure.

Definition 4.1. *Consider an ordered approximation space $(\Psi, \Re, \lesssim)$. For any subset $\Upsilon \subseteq \Psi$, we define the nano increasing topology with respect to Υ as $\tau^I_{\Re} = \Big\{\Psi, \emptyset, IL_{\Re}(\Upsilon), IU_{\Re}(\Upsilon), IB_{\Re}(\Upsilon)\Big\}$.*

The nano increasing topology $\tau^I_{\Re}$ is a topology on Ψ, satisfying the following axioms (rephrasing Proposition 3.1):

1. *Both Ψ and $\emptyset$ are elements of $\tau^I_{\Re}$.*
2. *The union of any subcollection of elements in $\tau^I_{\Re}$ is also an element of $\tau^I_{\Re}$.*
3. *The intersection of any finite subcollection of elements in $\tau^I_{\Re}$ is also an element of $\tau^I_{\Re}$.*

This nano increasing topology provides a way to define open sets on Ψ with respect to the equivalence relation $\Re$ and the partial order relation $\lesssim$. The open sets in $\tau^I_{\Re}$ are Ψ itself, the empty set $\emptyset$, the increasing lower approximation of Υ denoted by $IL_{\Re}(\Upsilon)$, the increasing upper approximation of Υ denoted by $IU_{\Re}(\Upsilon)$, and the increasing boundary approximation of the region of Υ denoted by $IB_{\Re}(\Upsilon)$.

Remark 4.1. *The nano decreasing topology with respect to a subset Υ of an ordered approximation space $(\Psi, \Re, \lesssim)$ is denoted by $\tau^D_{\Re}$ and defined as $\tau^D_{\Re} = \Big\{\Psi, \emptyset, DL_{\Re}(\Upsilon), DU_{\Re}(\Upsilon), DB_{\Re}(\Upsilon)\Big\}$.*

Example 4.1. *From Example 3.1, let $\Upsilon = \big\{\rho, \varsigma\big\} \subseteq \Psi$. Then:*
$IL_{\Re}(\Upsilon) = I\Re(\varsigma) \cap \Upsilon = \{\varsigma\}$, $IU_{\Re}(\Upsilon) = \big[DL_{\Re}(\Upsilon^c)\big]^c = \big[\{\delta\}\big]^c = \big\{\rho, \sigma, \varsigma\big\}$, *then*
$IB_{\Re}(\Upsilon) = IU_{\Re}(\Upsilon) - IL_{\Re}(\Upsilon) = \{\rho, \sigma, \varsigma\} - \{\varsigma\} = \{\rho, \sigma\}$. *Therefore,* $\tau^I_{\Re} = \Big\{\Psi, \emptyset, \{\varsigma\}, \{\rho, \sigma, \varsigma\}, \{\rho, \sigma\}\Big\}$.
And $DL_{\Re}(\Upsilon) = D\Re(\varsigma) \cap \Upsilon = \{\rho, \varsigma\}$, $DU_{\Re}(\Upsilon) = \big[IL_{\Re}(\Upsilon^c)\big]^c = [\{\delta\}]^c = \big\{\rho, \sigma, \varsigma\big\}$, *then*
$DB_{\Re}(\Upsilon) = DU_{\Re}(\Upsilon) - DL_{\Re}(\Upsilon) = \{\rho, \sigma, \varsigma\} - \{\rho, \varsigma\} = \{\sigma\}$. *Thus,* $\tau^D_{\Re} = \Big\{\Psi, \emptyset, \{\rho, \varsigma\}, \{\rho, \sigma, \varsigma\}, \{\sigma\}\Big\}$.

Definition 4.2. *Given a nano increasing topological space $(\Psi, \tau^I_{\Re_P})$, where $\Re_P$ is the equivalence relation with respect to the set of attributes P. Then, the degree of crispness of any subset $\Upsilon \subseteq \Psi$ is represented by a nano increasing accuracy measure $C^I_{\Re_P}(\Upsilon)$ and is defined as follows:*

$$C^I_{\Re_P}(\Upsilon) = \frac{|IL_{\Re_P}(\Upsilon)|}{|IU_{\Re_P}(\Upsilon)|}, \qquad \Upsilon \neq \emptyset.$$

It is clear that the value of $C^I_{\Re_P}(\Upsilon)$ is between 0 and 1. If $IL_{\Re_P}(\Upsilon) = IU_{\Re_P}(\Upsilon)$, then Υ is an increasing set. Otherwise, Υ is considered as an increasing rough set.

Note that: $C^D_{\Re_P}(\Upsilon) = \dfrac{|DL_{\Re_P}(\Upsilon)|}{|DU_{\Re_P}(\Upsilon)|}$, $\Upsilon \neq \emptyset$ is the nano decreasing accuracy measure.

5 Application of Nano Increasing (Decreasing) Topological Spaces

In this section, we present the concept of m-nano flou sets. Additionally, we provide an application of this concept to a real-life problem, where we compare our method with the previous method proposed by [16]. The comparison is done by calculating the accuracy measure for both methods.

Definition 5.1. *Let Ψ represent the universe, $\Re$ an equivalence relation on Ψ, and $\lesssim$ a partial order relation on Ψ. Suppose we have a set of attributes denoted as P, and Υ is a subset of Ψ. We define two topologies on Ψ with respect to Υ: $\tau^I_{\Re_P}(\Upsilon)$, which is the nano increasing topology, and $\tau^D_{\Re_P}(\Upsilon)$, which is the nano decreasing topology.*

An m-nano flou set ω of P, where $(m \geq 2)$, is described as an m-tuple. In this m-tuple, each component $\hbar_i$ is a subset of $\hbar_j$ if, for all $\hbar_i, \hbar_j \in P$, the following condition holds:

$$mid\Big(C^I_{\Re_{P-\{\hbar_i\}}}(\Upsilon), C^D_{\Re_{P-\{\hbar_i\}}}(\Upsilon)\Big) < mid\Big(C^I_{\Re_{P-\{\hbar_j\}}}(\Upsilon), C^D_{\Re_{P-\{\hbar_j\}}}(\Upsilon)\Big).$$

In this condition, $C^I_{\Re_{P-\hbar_i}}(\Upsilon)$ represents the nano increasing accuracy of Υ concerning the attributes P after removing $\hbar_i$, and $C^D_{Re_{P-\hbar_i}}(\Upsilon)$ represents the nano decreasing accuracy. The m-nano flou set ω is determined based on the comparison of these accuracy measures for all pairs of attributes $\hbar_i$ and $\hbar_j$ in P.

Algorithm

Option 1: Using the nano increasing topological space

Case 1: Patients with Heart Attack

Step 1: To represent the given information, follow these steps:

1. Begin with a finite universe Ψ.
2. Consider a finite set P of attributes, divided into two classes: P_1 for condition attributes and P_2 for the decision attribute.
3. Establish an equivalence relation $\Re$ on Ψ and a corresponding partial order relation $\lesssim$ based on P_1.
4. Take a subset Υ of Ψ.
5. Represent the data as an information table, where the columns are labeled by attributes from P, the rows are objects from Ψ, and the table entries represent the attribute values associated with each object.

Step 2: Calculate the increasing (decreasing) lower approximation, increasing (decreasing) upper approximation and the increasing (decreasing) boundary region of Υ with respect to $\Re_{P_1}$ and $\lesssim_{P_1}$.

Step 3: Construct the nano increasing (decreasing) topology $\tau^I_{\Re_{P_1}}(\tau^D_{\Re_{P_1}})$ on Ψ.

Step 4: Remove an attribute $\hbar$ from P_1 and find the increasing (decreasing) lower and upper approximations and the increasing (decreasing) boundary region of Υ with respect to $\Re_{P_1-\{\hbar\}}$ and $\lesssim_{P_1-\{\hbar\}}$, $\hbar \in P_1$.

Step 5: Generate the nano increasing (decreasing) topology $\tau^I_{\Re_{P_1-\{\hbar\}}}(\tau^D_{\Re_{P_1-\{\hbar\}}})$ on Ψ, $\hbar \in P_1$.

Step 6: Repeat steps 3 and 4 for all attributes in P_1.

Step 7: Attributes in P_1 for which $\hbar_i \subseteq \hbar_j,\ \forall i, j$ form the m-nano flou set of $\Re_{P_1}$.

Case 2: Patients not with Heart Attack.
Do the same steps.

Option 2: Using the nano decreasing topological space
Do the same steps in **Option 1**.

Example 5.1. *In this example, the objective is to find the key factors affecting "Heart Attack" using the nano increasing (decreasing) topology and topological reduction of attributes in an incomplete information system. The data set provided in Table 1 consists of information about patients related to factors such as High Blood Pressure, Alcohol and Smoking, Stress and Strain, Diabetics, and Family History.*

The set of patients being examined is represented by $\Psi = \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7, \flat_8\}$, and the set of factors under consideration is denoted by $P_1 = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5\}$, where $\hbar_1$ corresponds to High Blood Pressure, $\hbar_2$ to Alcohol and Smoking, $\hbar_3$ to Diabetics, $\hbar_4$ to Stress and Strain, and $\hbar_5$ to Family History and $P_2 = \{$ Result $\}$.

The equivalence relation $\Re$ on Ψ is defined as $\Re_{P_1} = \blacktriangle \cup \{(\flat_1, \flat_4), (\flat_4, \flat_1), (\flat_3, \flat_6), (\flat_6, \flat_3)\}$, which indicates that patients $\flat_1$ and $\flat_4$ are equivalent, as well as patients $\flat_3$ and $\flat_6$.

The partial order relation $\lesssim$, defined as $\lesssim \equiv \subset$, is given by $\lesssim_{P_1} = \blacktriangle \cup \{(\flat_5, \flat_1), (\flat_5, \flat_3), (\flat_5, \flat_4), (\flat_5, \flat_6)\}$. This implies that patient $\flat_5$ is related to patients $\flat_1$, $\flat_3$, $\flat_4$, and $\flat_6$.

Using these relations, the nano increasing (decreasing) topology can be applied to identify the key factors influencing "Heart Attack" by performing topological reduction of attributes in the incomplete information system.

In Table 1, we have two different options to analyze the data:

Table 1: Tabular information about patients those who are having high blood pressure, alcohol and smoking, stress and strain, diabetics and family history [16].

Objects	$\hbar_1$	$\hbar_2$	$\hbar_3$	$\hbar_4$	$\hbar_5$	Result
$\flat_1$	Yes	Yes	Yes	No	No	$\surd$
$\flat_2$	Yes	No	No	Yes	Yes	$\surd$
$\flat_3$	No	Yes	Yes	No	Yes	$\surd$
$\flat_4$	Yes	Yes	Yes	No	No	Nil
$\flat_5$	No	Yes	Yes	No	No	Nil
$\flat_6$	No	Yes	Yes	No	Yes	Nil
$\flat_7$	Yes	No	Yes	No	Yes	$\surd$
$\flat_8$	Yes	No	Yes	Yes	No	Nil

Option 1: Using the nano increasing topological space:

Case 1: (Patients with Heart Attack)
Let $\Upsilon = \{\flat_1, \flat_2, \flat_3, \flat_7\}$, the set of patient with Heart Attack.

Then, $IL_{\Re_{P_1}}(\Upsilon) = \{\flat_2, \flat_7\}$, $IU_{\Re}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}$ and $IB_{\Re_{P_1}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_6\}$.
Therefore, $\tau^I_{\Re_{P_1}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 1: After removing the attribute "$\hbar_1 =$ High Blood Pressure" from P_1,
$\Re_{P_1-\{\hbar_1\}} = \blacktriangle \cup \Big\{(\flat_1, \flat_4), (\flat_4, \flat_1), (\flat_1, \flat_5), (\flat_5, \flat_1), (\flat_5, \flat_4), (\flat_4, \flat_5), (\flat_3, \flat_6), (\flat_6, \flat_3)\Big\}$ and
$\precsim_{P_1-\{\hbar_1\}} = \blacktriangle \cup \Big\{(\flat_1, \flat_3), (\flat_1, \flat_6), (\flat_4, \flat_3), (\flat_5, \flat_3), (\flat_7, \flat_3), (\flat_4, \flat_6), (\flat_5, \flat_6), (\flat_7, \flat_6)\Big\}$.
Then, $IL_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_2, \flat_3, \flat_7\}$, $IU_{\Re_{P-\{\hbar_1\}}}(\Upsilon) = \Big\{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\Big\}$ and
$IB_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_1, \flat_4, \flat_5, \flat_6\} \nsubseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_1\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_3, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 2: After removing the attribute "$\hbar_2 =$ Alcohol and smoking" from P_1,
$\Re_{P_1-\{\hbar_2\}} = \blacktriangle \cup \Big\{(\flat_1, \flat_4), (\flat_4, \flat_1), (\flat_3, \flat_6), (\flat_6, \flat_3)\Big\}$ and
$\precsim_{P_1-\{\hbar_2\}} = \blacktriangle \cup \Big\{(\flat_5, \flat_1), (\flat_1, \flat_7), (\flat_1, \flat_8), (\flat_5, \flat_3), (\flat_3, \flat_7), (\flat_5, \flat_4), (\flat_4, \flat_7), (\flat_4, \flat_8),$
$(\flat_5, \flat_6), (\flat_5, \flat_7), (\flat_5, \flat_8), (\flat_6, \flat_7)\Big\}$.
Then, $IL_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $IU_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_6, \flat_7\}$ and
$IB_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_6\} \subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_2\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 3: After removing the attribute "$\hbar_3 =$ Diabetics" from P_1,
$\Re_{P_1-\{\hbar_3\}} = \blacktriangle \cup \Big\{(\flat_1, \flat_4), (\flat_4, \flat_1), (\flat_3, \flat_6), (\flat_6, \flat_3)\Big\}$ and
$\precsim_{P_1-\{\hbar_3\}} = \blacktriangle \cup \Big\{(\flat_5, \flat_1), (\flat_7, \flat_2), (\flat_8, \flat_2), (\flat_5, \flat_3), (\flat_5, \flat_4), (\flat_5, \flat_6)\Big\}$.
Then, $IL_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $IU_{R_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}$ and
$IB_{\Re_{P-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_6\} \subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_3\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 4: After removing the attribute "$\hbar_4 =$ Stress and strain" from P_1,
$\Re_{P_1-\{\hbar_4\}} = \blacktriangle \cup \Big\{(\flat_1, \flat_4), (\flat_4, \flat_1), (\flat_3, \flat_6), (\flat_6, \flat_3)\Big\}$ and
$\precsim_{P_1-\{\hbar_4\}} = \blacktriangle \cup \Big\{(\flat_5, \flat_1), (\flat_8, \flat_1), (\flat_2, \flat_7), (\flat_5, \flat_3), (\flat_5, \flat_4), (\flat_8, \flat_4), (\flat_5, \flat_6), (\flat_8, \flat_7)\Big\}$.
Then, $IL_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $IU_{\Re_{P-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}$ and
$IB_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_6\} \subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_4\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 5: After removing the attribute "$\hbar_5 =$ Family history" from P_1,
$\Re_{P_1-\{\hbar_5\}} = \blacktriangle \cup \Big\{(\flat_1, \flat_4), (\flat_4, \flat_1), (\flat_3, \flat_5), (\flat_5, \flat_3), (\flat_5, \flat_6), (\flat_6, \flat_5), (\flat_3, \flat_6), (\flat_6, \flat_3)\Big\}$ and
$\precsim_{P_1-\{\hbar_5\}} = \blacktriangle \cup \Big\{(\flat_3, \flat_1), (\flat_5, \flat_1), (\flat_7, \flat_1), (\flat_2, \flat_8), (\flat_3, \flat_4), (\flat_5, \flat_4), (\flat_6, \flat_4),$
$(\flat_7, \flat_4), (\flat_7, \flat_8)\Big\}$.
Then $IL_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $IU_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\}$ and
$IB_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\} \nsubseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_5\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}\Big\}$.

Case 2: (Patients not with Heart Attack)
Let $\Upsilon = \{\flat_4, \flat_5, \flat_6, \flat_8\}$, the set of patient without Heart Attack.
Then, $IL_{\Re_{P_1}}(\Upsilon) = \{\flat_4, \flat_5, \flat_6, \flat_8\}$, $IU_{\Re_{P_1}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$
and $IB_{\Re_{P_1}}(\Upsilon) = \{\flat_1, \flat_3\}$.
Therefore, $\tau^I_{\Re_{P_1}} = \Big\{\Psi, \emptyset, \{\flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3\}\Big\}$.

Step 1: After removing the attribute "$\hbar_1$ = High Blood Pressure" from P_1.
Then, $IL_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_8\}$, $IU_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $IB_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\} \not\subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_1\}}} = \Big\{\Psi, \emptyset, \{\flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}\Big\}$.

Step 2: After removing the attribute "$\hbar_2$ = Alcohol and smoking" from P_1.
Then, $IL_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = IU_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \{\flat_4, \flat_5, \flat_6, \flat_8\}$ and $IB_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \emptyset \subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_2\}}} = \Big\{\Psi, \emptyset, \{\flat_4, \flat_5, \flat_6, \flat_8\}\Big\}$.

Step 3: After removing the attribute "$\hbar_3$ = Diabetics" from P_1.
Then, $IL_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_4, \flat_5, \flat_6, \flat_8\}$, $IU_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $IB_{\Re_{P-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_3\} \subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_3\}}} = \Big\{\Psi, \emptyset, \{\flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3\}\Big\}$.

Step 4: After removing the attribute "$\hbar_4$ = Stress and strain" from P_1.
Then, $IL_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_4, \flat_5, \flat_6, \flat_8\}$, $IU_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $IB_{\Re_{P-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_3\} \subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_4\}}} = \Big\{\Psi, \emptyset, \{\flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3\}\Big\}$.

Step 5: After removing the attribute "$\hbar_5$ = Family history" from P_1.
Then, $IL_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_8\}$, $IU_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $IB_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\} \not\subseteq IB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^I_{\Re_{P_1-\{\hbar_5\}}} = \Big\{\Psi, \emptyset, \{\flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}\Big\}$.

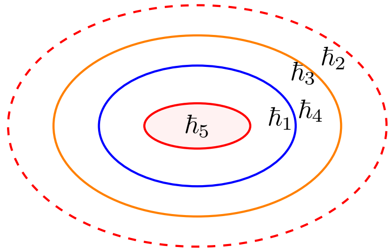
Now we generate the m-nano flou set on P_1 in the case of patients with heart attack:

1. $mid\Big(C^I_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon)\Big) = \dfrac{\frac{3}{7} + \frac{2}{7}}{2} = \dfrac{2.5}{7}$.
2. $mid\Big(C^I_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon)\Big) = \dfrac{\frac{2}{5} + 1}{2} = \dfrac{3.5}{5}$.
3. $mid\Big(C^I_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon)\Big) = \dfrac{\frac{1}{3} + \frac{1}{2}}{2} = \dfrac{2.5}{6}$.
4. $mid\Big(C^I_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon)\Big) = \dfrac{\frac{1}{3} + \frac{1}{2}}{2} = \dfrac{2.5}{6}$.
5. $mid\Big(C^I_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon)\Big) = \dfrac{\frac{2}{7} + \frac{2}{7}}{2} = \dfrac{2}{7}$.

Clear that: m-nano flou set $= (\hbar_5, \hbar_1, \hbar_3 = \hbar_4, \hbar_2)$.

Remark 5.1.

1. $mid\Big(C^I_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon)\Big) = mid\Big(C^I_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon)\Big) = mid\Big(C^I_{\Re_{P_1}}(\Upsilon), C^D_{\Re_{P_1}}(\Upsilon)\Big) = \dfrac{2.5}{6}$.
2. *The core of this m-nano flou set is $\hbar_5$.*
3. *The attribute $\hbar_2$ doesn't affect the results.*



Option 2: Using the nano decreasing topological space:

Case 1: (Patients with Heart Attack)
Let $\Upsilon = \{\flat_1, \flat_2, \flat_3, \flat_7\}$, the set of patient with Heart Attack.
Then, $DL_{\Re_{P_1}}(\Upsilon) = \{\flat_2, \flat_7\}$, $DU_{\Re_{P_1}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_7\}$ and $DB_{\Re_{P_1}}(\Upsilon) = \{\flat_1, \flat_3\}$.
Therefore, $\tau^D_{\Re_{P_1}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_7\}, \{\flat_1, \flat_3\}\Big\}$.

Step 1: After removing the attribute "$\hbar_1$ = High blood pressure" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $DU_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\}$ and $DB_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\} \nsubseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_1\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}\Big\}$.

Step 2: After removing the attribute "$\hbar_2$ = Alcohol and smoking" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = DU_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_7\}$ and $DB_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \emptyset \subseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_2\}}} = \Big\{\Psi, \emptyset, \{\flat_1, \flat_2, \flat_3, \flat_7\}\Big\}$.

Step 3: After removing the attribute "$\hbar_3$ = Diabetics" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $DU_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_7\}$ and $DB_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_3\} \subseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_3\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_7\}, \{\flat_1, \flat_3\}\Big\}$.

Step 4: After removing the attribute "$\hbar_4$ = Stress and strain" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $DU_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_7\}$ and $DB_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_3\} \subseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_4\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_7\}, \{\flat_1, \flat_3\}\Big\}$.

Step 5: After removing the attribute "$\hbar_5$ = Family history" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_2, \flat_7\}$, $DU_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\}$ and $DB_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\} \nsubseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_5\}}} = \Big\{\Psi, \emptyset, \{\flat_2, \flat_7\}, \{\flat_1, \flat_2, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}\Big\}$.

Case 2: (Patients not with Heart Attack)
Let $\Upsilon = \{\flat_4, \flat_5, \flat_6, \flat_8\}$, the set of patient without Heart Attack.
Then, $DL_{\Re_{P_1}}(\Upsilon) = \{\flat_5, \flat_8\}$, $DU_{\Re_{P_1}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $DB_{\Re_{P_1}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_6\}$.
Therefore, $\tau^D_{\Re_{P_1}} = \Big\{\Psi, \emptyset, \{\flat_5, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 1: After removing the attribute "$\hbar_1$ = High blood pressure" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_8\}$, $DU_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_1, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $DB_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon) = \{\flat_1, \flat_4, \flat_5, \flat_6\} \nsubseteq DB_{R_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_1\}}} = \Big\{\Psi, \emptyset, \{\flat_8\}, \{\flat_1, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_4, \flat_5, \flat_6\}\Big\}$.

Step 2: After removing the attribute "$\hbar_2 =$ Alcohol and smoking" from P_1.
Then, $DL_{\Re_{P_1-\hbar_2}}(\Upsilon) = \{\flat_1, \flat_4, \flat_5, \flat_8\}$, $DU_{\Re_{P_1-\hbar_2}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $DB_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon) = \{\flat_3, \flat_6\} \subseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_2\}}} = \Big\{\Psi, \emptyset, \{\flat_1, \flat_4, \flat_5, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_3, \flat_6\}\Big\}$.

Step 3: After removing the attribute "$\hbar_3 =$ Diabetics" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_5, \flat_8\}$, $DU_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $DB_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_6\} \subseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_3\}}} = \Big\{\Psi, \emptyset, \{\flat_5, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 4: After removing the attribute "$\hbar_4 =$ Stress and strain" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_5, \flat_8\}$, $DU_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $DB_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon) = \{\flat_1, \flat_3, \flat_4, \flat_6\} \subseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_4\}}} = \Big\{\Psi, \emptyset, \{\flat_5, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_1, \flat_3, \flat_4, \flat_6\}\Big\}$.

Step 5: After removing the attribute "$\hbar_5 =$ Family history" from P_1.
Then, $DL_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_8\}$, $DU_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$ and $DB_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon) = \{\flat_3, \flat_4, \flat_5, \flat_6\} \not\subseteq DB_{\Re_{P_1}}(\Upsilon)$.
Therefore, $\tau^D_{\Re_{P_1-\{\hbar_5\}}} = \Big\{\Psi, \emptyset, \{\flat_8\}, \{\flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}, \{\flat_3, \flat_4, \flat_5, \flat_6\}\Big\}$.

Now we generate the m-nano flou set on P_1 in the case of patients without heart attack:

1. $mid\Big(C^I_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_1\}}}(\Upsilon)\Big) = \dfrac{\frac{1}{6} + \frac{1}{5}}{2} = \dfrac{5.5}{30}$.
2. $mid\Big(C^I_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_2\}}}(\Upsilon)\Big) = \dfrac{1 + \frac{2}{3}}{2} = \dfrac{2.5}{3}$.
3. $mid\Big(C^I_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon)\Big) = \dfrac{\frac{2}{3} + \frac{1}{3}}{2} = \dfrac{1}{2}$.
4. $mid\Big(C^I_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon)\Big) = \dfrac{\frac{2}{3} + \frac{1}{3}}{2} = \dfrac{1}{2}$.
5. $mid\Big(C^I_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_5\}}}(\Upsilon)\Big) = \dfrac{\frac{1}{6} + \frac{1}{5}}{2} = \dfrac{5.5}{30}$.

Clear that: m-nano flou set $= (\hbar_1 = \hbar_5, \hbar_3 = \hbar_4, \hbar_2)$.

Remark 5.2.

1. $mid\Big(C^I_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_3\}}}(\Upsilon)\Big) = mid\Big(C^I_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon), C^D_{\Re_{P_1-\{\hbar_4\}}}(\Upsilon)\Big) = mid\Big(C^I_{\Re_{P_1}}(\Upsilon), C^D_{\Re_{P_1}}(\Upsilon)\Big) = \dfrac{1}{2}$.
2. *The core of this m-nano flou set is $\hbar_1, \hbar_5$.*
3. *The attribute $\hbar_2$ doesn't affect the results.*

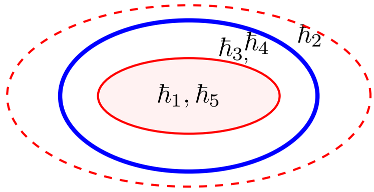


Table 2: Comparison of boundary and accuracy between Jayalakshmi method [16] and the current method (Patient with Heart Attack) in Definition 4.1 using Example 5.1. (Case (1)).

X	Jayalakshmi method [16]		Option (1) method		Option (2) method	
	$B_{\Re_X}(\Upsilon)$	$C_{\Re_X}(\Upsilon)$	$IB_{\Re_X}(\Upsilon)$	$C^I_{\Re_X}(\Upsilon)$	$DB_{\Re_X}(\Upsilon)$	$C^D_{\Re_X}(\Upsilon)$
P_1	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3\}$	$\frac{1}{2}$
$P_1 - \{\hbar_1\}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{2}{7}$	$\{\flat_1, \flat_4, \flat_5, \flat_6\}$	$\frac{3}{7}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{2}{7}$
$P_1 - \{\hbar_2\}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3, \flat_6\}$	$\frac{2}{5}$	$\emptyset$	1
$P_1 - \{\hbar_3\}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3\}$	$\frac{1}{2}$
$P_1 - \{\hbar_4\}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3\}$	$\frac{1}{2}$
$P_1 - \{\hbar_5\}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{2}{7}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{2}{7}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{2}{7}$

Table 3: Comparison of boundary and accuracy between Jayalakshmi method [16] and the current method (Patient not with Heart Attack) in Definition 4.1 using Example 5.1. (Case (2)).

X	Jayalakshmi method [16]		Option (1) method		Option (2) method	
	$B_{\Re_X}(\Upsilon)$	$C_{\Re_X}(\Upsilon)$	$IB_{\Re_X}(\Upsilon)$	$C^I_{\Re_X}(\Upsilon)$	$DB_{\Re_X}(\Upsilon)$	$C^D_{\Re_X}(\Upsilon)$
P_1	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{6}$	$\{\flat_1, \flat_3\}$	$\frac{2}{3}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$
$P_1 - \{\hbar_1\}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{1}{6}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{1}{6}$	$\{\flat_1, \flat_4, \flat_5, \flat_6\}$	$\frac{1}{5}$
$P_1 - \{\hbar_2\}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\emptyset$	1	$\{\flat_3, \flat_6\}$	$\frac{2}{3}$
$P_1 - \{\hbar_3\}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3\}$	$\frac{2}{3}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$
$P_1 - \{\hbar_4\}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$	$\{\flat_1, \flat_3\}$	$\frac{2}{3}$	$\{\flat_1, \flat_3, \flat_4, \flat_6\}$	$\frac{1}{3}$
$P_1 - \{\hbar_5\}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_7\}$	$\frac{1}{7}$	$\{\flat_1, \flat_3, \flat_4, \flat_5, \flat_6, \flat_8\}$	$\frac{1}{6}$	$\{\flat_3, \flat_4, \flat_5, \flat_6\}$	$\frac{1}{5}$

Based on our previous discussion, the present methodology proves to be more appropriate when compared to the approach proposed by Jayalakshmi in 2017 [16]. This is supported by the following findings we have established:

1. The factor $\hbar_5$ holds the highest level of influence.
2. The factor $\hbar_2$ demonstrates no discernible impact.
3. We have determined the relative significance of each key factor on the occurrence of "Heart Attack".

4. We have identified the key factors that exhibit an equivalent level of impact on "Heart Attack".

Remark 5.3. *In the example, the tables (Table 2, Table 3) show cases the results of the analysis based on the degree of crispness $C^{I}_{\Re_{P_1}}(\Upsilon)$ for each factor in the set P_1. The remark specifically focuses on the vertical comparison of $C^{I}_{\Re_{P_1-\{\hbar\}}}(\Upsilon)$ and $C^{D}_{\Re_{P_1-\{\hbar\}}}(\Upsilon)$ values, $\forall\, \hbar \in P_1$.*

For instance, it is noted that $C^{I}_{\Re_{P_1}}(\Upsilon)$ is $\dfrac{2}{3}$ in Table 3. However, when high blood pressure is removed (deleted), the value decreases to $\dfrac{1}{6}$. This decrease in the degree of crispness suggests that High Blood Pressure is a significant key factor affecting "Heart Attack".

Observation:

Based on the information presented in the previous tables (Table 2, Table 3), we can observe the relative order of the key factors' effects on Heart Attack. The order of influence, from the most significant to the least significant, is as follows: Family history, high blood pressure, diabetics, stress and strain, and finally alcohol and smoking.

This observation suggests that family history has the strongest influence on the occurrence of "Heart Attack", followed by high blood pressure, diabetics, stress and strain, and alcohol and smoking, which has the least discernible impact. By comparing the degree of crispness for each factor, we can determine the relative importance and contribution of these factors towards the occurrence of Heart Attack. Family history emerges as the most influential factor, while alcohol and smoking exhibit minimal or negligible influence.

This rephrasing summarizes the relative order of influence of the key factors on "Heart Attack" based on the analysis performed using the nano increasing (decreasing) topological space approach.

6 Conclusion

Rough set theory, originally introduced by Pawlak [30], has laid the fundamental groundwork for advancing data analysis in diverse fields. Building upon this foundation, our research introduces a novel dimension through the utilization of nano ordered topological spaces. In doing so, we extend the work of Lellis and Thivagar on nano topological spaces [35] and incorporate insights from Jayalakshmi's illuminating case study [16]. Together, these contributions have enabled us to gain a deeper understanding of the risk factors associated with heart failure, including high blood pressure, family history, and increased stress.

It is essential to acknowledge certain limitations that have shaped our research journey. Foremost among these is the constraint imposed by the size of our research sample. While our findings are promising and conceptually sound, they are based on a relatively limited dataset. Expanding our investigations to encompass a more extensive and diverse range of cases would bolster the robustness of our conclusions.

Additionally, our work is rooted in the development and formalism of our extended rough set model. Its real-world application to extensive medical datasets is an area ripe for future exploration. To truly gauge the practical benefits and utility of our model, comprehensive empirical studies and clinical validations are imperative. The potential for our research to evolve and make a lasting impact is substantial. As we look ahead, several avenues for future work emerge:

Conducting extensive empirical studies on larger and more varied medical datasets can substantiate the practical benefits of our extended rough set model. Real-world validation is a crucial step toward translating theoretical advancements into tangible clinical applications.

Future research can delve deeper into fine-tuning the parameters of our novel constructs, such as nano ordered topological spaces. Optimization for specific medical contexts can enhance the precision of our analytical methodologies.

The integration of our model with artificial intelligence techniques, including machine learning and deep learning, holds great promise. Such integration can lead to further improvements in predictive accuracy for heart failure diagnosis and prognosis. Collaborations with healthcare institutions and clinicians to conduct clinical trials can provide invaluable insights into the practical utility of our methodology. These trials can also guide the customization of our model to meet the needs of the medical community.

Expanding the scope of our research to encompass interdisciplinary collaborations can lead to innovative approaches in addressing heart failure, leveraging expertise from genetics, cardiology, data science, and beyond. In future work, we plan to explore applications in decision-making by utilizing the properties of nano bi-ordered topological space.

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Conflicts of Interest The authors declare no conflict of interest.

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